Self – similar solutions of the Burgers hierarchy

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Abstract

Self — similar solutions of the equations for the Burgers hierarchy are presented.

1 Introduction

The Burgers hierarchy can be written in the form [1–4]

$$u_t + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + u\right)^n u = 0, \quad n = 0, 1, 2, \ldots. \tag{1}$$

Assuming $n = 1$ in Eq. (1) we have the Burgers equation

$$u_t + 2\,u\,u_x + u_{xx} = 0. \tag{2}$$

Eq. (2) was firstly introduced in [5]. It is well known that this equation can be linearized by means of the Cole-Hopf transformation [6–8]. Exact solutions of Eq.(2) were considered in many papers (see, for example, [9–12]).

Assuming $n = 2$ in Eq. (1) we obtain the Sharma - Tasso - Olver equation

$$u_t + u_{xxx} + 3\,u_x^2 + 3\,u\,u_{xx} + 3\,u^2\,u_x = 0. \tag{3}$$

The Sharma - Tasso - Olver equation was derived in [1, 13]. Some exact solutions of this equation were presented in [14–21].

At $n = 3$ and $n = 4$ we obtain the following fourth and fifth order partial differential equations

$$\begin{aligned} u_t + u_{xxxx} + 10\,u_x u_{xx} + 4\,u u_{xxx} + 12\,u u_x^2 + \\ +6\,u^2 u_{xx} + 4\,u^3\,u_x = 0, \end{aligned} \tag{4}$$

$$
\begin{gathered}
u_t + u_{xxxxxx} + 10\,u_{xx}^2 + 15\,u_x u_{xxx} + 5\,u u_{xxxxx} + 15\,u_x^3 + \\
+50\,u u_x u_{xx} + 10\,u^2 u_{xxx} + 30\,u^2 u_x^2 + 10\,u^3 u_{xx} + 5\,u^4 u_x = 0.
\end{gathered} \tag{5}
$$

Assuming

$$
x = L\,x', \quad u = C_0\,u', \quad t = T\,t', \tag{6}
$$

we have that Eq.(1) is invariant under the dilation group in the case

$$
C_0\,L = 1, \quad T = L^{n+1}. \tag{7}
$$

Assuming $C_0 = e^{-a}$ in (7), we obtain the delation group for the Burgers hierarchy (1) in the form

$$
u' = e^{-a}\,u, \quad x' = e^{a}\,x, \quad t' = e^{a(n+1)}\,t. \tag{8}
$$

From transformations (8) we have two invariants for Eq.(1)

$$
I_1 = u\,t^{\frac{1}{n+1}} = u'\,(t')^{\frac{1}{n+1}}, \qquad I_2 = \frac{x}{t^{\frac{1}{n+1}}} = \frac{x'}{(t')^{\frac{1}{n+1}}}. \tag{9}
$$

Therefore we look for the solutions of the Burgers hierarchy taking into account the variables

$$
u(x,t) = \frac{A}{t^{\frac{1}{n+1}}}\,f(z), \quad z = \frac{B\,x}{t^{\frac{1}{n+1}}}. \tag{10}
$$

Substituting (10) into (1) we obtain the equation for $f(z)$ at

$$
A = B = \frac{1}{(n+1)^{\frac{1}{n+1}}}. \tag{11}
$$

in the form

$$
\left(\frac{d}{dz} + f\right)^n f - z\,f + \beta = 0, \tag{12}
$$

where β is the constant of integration.

Solving Eq.(12) we obtain solutions of the Burgers hierarchy in the form

$$
u(x,t) = \frac{1}{(n\,t + t)^{\frac{1}{n+1}}} f(z), \quad z = \frac{x}{(n\,t + t)^{\frac{1}{n+1}}}. \tag{13}
$$

Let us study the solutions of nonlinear ordinary differential equation (12).

2 Exact solutions of equation(12)

First of all let us prove the following lemma.

Lemma 1. *Equation* (12) *can be transformed to the linear equation of* $(n+1)$ *- th order by means of transformation*

$$f = \frac{\psi_z}{\psi}. \tag{14}$$

Proof. The proof of this lemma can be given by means of the mathematical induction method.

Using the transformation (14) we have

$$\left(\frac{d}{dz} + f\right) f = \frac{\psi_{zz}}{\psi}, \quad \left(\frac{d}{dz} + f\right)^2 f = \frac{\psi_{zzz}}{\psi} \tag{15}$$

Assuming that there is equality

$$\left(\frac{d}{dz} + f\right)^k f = \frac{\psi_{k+1,z}}{\psi}, \quad \psi_{k+1,z} = \frac{d^{k+1}\psi}{d\,z^{k+1}}. \tag{16}$$

Differentiating Eq.(16) with respect to in z we have

$$\frac{d}{dz}\left(\frac{d}{dz} + f\right)^k f = \frac{\psi_{k+2,z}}{\psi} - \frac{\psi_z\,\psi_{k+1,z}}{\psi^2}. \tag{17}$$

From Eq.(17) we obtain the equality

$$\left(\frac{d}{dz} + f\right)^{k+1} f = \frac{\psi_{k+2,z}}{\psi}. \tag{18}$$

Therefore we obtain the formula

$$\left(\frac{d}{dz} + f\right)^n f = \frac{\psi_{n+1,z}}{\psi}. \tag{19}$$

Taking this formula into account we have the equality

$$\left(\frac{d}{dz} + f\right)^n f - z\,f + \beta = \frac{1}{\psi}\,(\psi_{n+1,z} - z\,\psi_z + \beta\,\psi). \tag{20}$$

As result of this lemma we obtain that solutions of Eq. (12) can be found by the formula (14), where $\psi(z)$ is the solution of the linear equation

$$\psi_{n+1,z} - z\,\psi_z + \beta\,\psi = 0, \tag{21}$$

Let us consider the partial cases. Assuming $\beta = 0$ in Eq.(21) we have

$$\psi_{n+1,z} - z\,\psi_z = 0. \tag{22}$$

Denoting $\psi_z = y$ we obtain

$$y_{n,z} - z\,y = 0. \tag{23}$$

In the case $n = 1$ we get solution of Eq.(23) in the form

$$y(z) = C_2\, e^{-\frac{z^2}{2}}. \tag{24}$$

The general solution of Eq.(23) can be written as

$$\psi(z) = C_3 + C_2 \int_0^z e^{-\frac{\xi^2}{2}} d\xi, \tag{25}$$

where C_2 and C_3 are arbitrary constants. In the case $n = 2$ we obtain the general solution of Eq.(23) in the form

$$y(z) = C_4\,\sqrt{z} J_{\frac{1}{3}} \left(\frac{2}{3}\, z^{\frac{3}{2}}\right) + C_5\,\sqrt{z}\, Y_{\frac{1}{3}} \left(\frac{2}{3}\, z^{\frac{3}{2}}\right), \tag{26}$$

where $J_{\frac{1}{3}}$ and $Y_{\frac{1}{3}}$ are the Bessel functions.

In the case $n > 2$ solution of Eq.(23) has n solutions

$$y_j(z) = z^{j-1}\, E_{n,1+\frac{1}{n},1+\frac{j}{n}}(z^{n+1}), \quad j = 1, 2, \ldots, n, \tag{27}$$

where $E_{n,m,l}$ is a Mittag - Leffler type special function defined by [22];

$$E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} b_k\, z^k, \quad b_k = \prod_{s=0}^{k-1} \frac{\Gamma\left(n(ms+l)+1\right)}{\Gamma(n(ms+l+1)+1)} \tag{28}$$

In the case $\beta \neq 0$ solutions of Eq.(23) can be referred to the type of the Laplace equations [23]. There are partial solutions $\psi(z) = -z^m$ of Eq.(21) at $\beta = m$, where $0 < m \leq n$ is integer. In the general case solutions of equations (21) can be found using the Laplace transformation or taking the expansions in the power series into account.

For a example let us solve the Cauchy problem for linear ordinary differential equation (21) at $\beta = -1$. We have the following problem

$$\begin{gathered} \psi_{n+1,z} - z\,\psi_z - \,\psi = 0, \\ \psi(z=0) = b_0, \quad \psi_z(z=0) = b_1, \ldots, \psi_{n-2,z} = b_{n-2} \quad \psi_{n-1,z} = b_{n-1}. \end{gathered} \tag{29}$$

Substituting

$$\psi(z) = \sum_{m=0}^{\infty} a_m\, z^m \tag{30}$$

into Eq.(29), we obtain the solution in the form

$$\begin{aligned}
\psi(z) = a_0 \sum_{k=0}^{\infty} \frac{z^{nk}\, \prod_{j=0}^{k} (n\,j+1)}{(nk+1)!} + a_1 \sum_{k=0}^{\infty} \frac{z^{nk+1}\, \prod_{j=0}^{k} (n\,j+2)}{(nk+2)!} + \\
+2\, a_2 \sum_{k=0}^{\infty} \frac{z^{nk+2}\, \prod_{j=0}^{k} (n\,j+3)}{(nk+3)!} + \ldots + \\
+(n-2)!\, a_{n-2} \sum_{k=0}^{\infty} \frac{z^{nk+n-2}\, \prod_{j=0}^{k} (n\,j+n-1)}{(nk+n-1)!} + \\
+(n-1)!\, a_{n-1} \sum_{k=0}^{\infty} \frac{z^{nk+n-1}\, \prod_{j=0}^{k} (n\,j+n)}{(nk+n)!}.
\end{aligned} \tag{31}$$

The value of coefficients $a_0,\, a_1,\, a_2, \ldots, a_{n-2}$ and a_{n-1} are determined by the initial values $b_0,\, b_1,\, b_2, \ldots, b_{n-2}$ and b_{n-1}. We have

$$a_0 = b_0, \quad a_1 = b_1, \quad a_2 = \frac{b_2}{(2!)^2}, \ldots, a_{n-1} = \frac{b_{n-1}}{((n-1)!)^2}. \tag{32}$$

Let us present the partial cases of solution for equation (29). In the case $n = 3$ we have solution in the form

$$\begin{aligned}
\psi(z) = a_0 \sum_{k=0}^{\infty} \frac{z^{3k}\, \prod_{j=0}^{k} (3\,j+1)}{(3k+1)!} + a_1 \sum_{k=0}^{\infty} \frac{z^{3k+1}\, \prod_{j=0}^{k} (3\,j+2)}{(3k+2)!} + \\
+2\, a_2 \sum_{k=0}^{\infty} \frac{z^{3k+2}\, \prod_{j=0}^{k} (3\,j+3)}{(3k+3)!}.
\end{aligned} \tag{33}$$

Assuming $n=4$ we obtain

$$\psi(z)=a_0\sum_{k=0}^{\infty}\frac{z^{4k}\,\prod_{j=0}^{k}(4\,j+1)}{(4k+1)!}+a_1\sum_{k=0}^{\infty}\frac{z^{4k+1}\,\prod_{j=0}^{k}(4\,j+2)}{(4k+2)!}+$$
$$+2\,a_2\sum_{k=0}^{\infty}\frac{z^{4k+2}\,\prod_{j=0}^{k}(4\,j+3)}{(4k+3)!}+6\,a_3\sum_{k=0}^{\infty}\frac{z^{4k+3}\,\prod_{j=0}^{k}(4\,j+4)}{(4k+4)!}. \tag{34}$$

One can show that these power series are conversed for any values z. Therefore self-similar solutions of equations for the Burgers hierarchy are found after substitution (34) into formula (14).

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